ANALYTIC SOLUTION OF THE UNSTEADY INVERSE HEAT-CONDUCTION PROBLEM

P. I. Balk

UDC 526.24.02

An algorithm is given for an approximate analytic solution of the unsteady inverse heat-condue-tion problem for infinite and semi-infinite objects. The solution found in this manner is used to solve a special inverse problem of free diffusion.

The number of heat-conduction problems, both forward and inverse, which can be solved in finite form is extremely limited. This circumstance prevents the widespread use of analytic methods for approximately solving the inverse problem; in certain cases these methods greatly simplify the calculation process. Consequently, there is definite interest in searching for new exact solutions of the inverse problems.

In the present paper we are concerned with the analytic solution of two inverse problems. The first deals with the classical type of inverse heat-conduction problems [2]; the second arises in the solution of problems of diffuse scattering and is of a specialized nature. However, by virtue of the analogy between heat conduction and diffusion, the results can be automatically applied to the similar heat-conduction problem.

For simplicity we restrict the analysis to the linear case. We first outline the analytic method for approximately finding the temperature $T(x, \tau_0) = f(x)$ within an infinite heat-insulating rod at any time $\tau_0 < \tau$ on the basis of a specified temperature distribution $T(x, \tau)$ at time τ, assuming that the thermal diffusivity a is a known constant and that there are no heat sources. For this purpose we first examine the corresponding direct problem: that of determining the temperature $T(x, \tau)$ at time τ from the known temperature distribution at an earlier time τ_0 (without any loss of generality we can assume $\tau = 0$ as we do below). The solution of this problem, according to [3], is given by the integral equation

$$T(x, \tau) = \frac{1}{\sqrt{4\pi a\tau}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4a\tau}\right] d\xi. \tag{1}$$

An approximate solution of this problem can be constructed with the help of a convenient approximation of the function $f(\xi)$:

$$f(\xi) = \sum_{n=0}^{L} A_n \psi_n(\xi). \tag{2}$$

Now (1) is integrable in terms of elementary functions. From the mathematical standpoint the system $\Psi = \{\psi_i(\xi)\}_{i=0}^{\infty}$ should be chosen such that the function in (2) can be used, with an appropriate choice of the number L of real parameters A_n in this faction and with an appropriate choice of the values of these parameters, to achieve an arbitrarily exact description of any temperature distribution $f(\xi)$. This requirement is satisfied, e.g., by the system of algebraic monomials

$$\Psi = \{\xi^i\}_{i=0}^{\infty}, \tag{3}$$

whose linear shells

$$P(\xi) = \sum_{n=0}^{L} A_n \xi^n \tag{4}$$

Institute of Geochemistry, Siberian Branch, Academy of Sciences of the USSR, Irkutsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 28, No. 4, pp. 716-722, April, 1975. Original article submitted August 29, 1974.

give a uniform approximation of any continuous function. Assuming $f(\xi) = P(\xi)$ and using the tabulated equations of [4], we write the integral representation in (1) and the function T in the finite form

$$T_P(x,\ \tau) = T(x,\ \tau)|_{f=P} = \sum_{n=0}^{L} A_n \varphi_n(x,\ \sigma), \tag{5}$$

where

$$\varphi_n(x,\ \sigma) = \sum_{i=0}^{\lambda} (\delta^n + 2i)(2\lambda - 2i - 1)!!\sigma^{2(\lambda-i)} x^{\delta+2i}; \tag{6}$$

$$\delta = \begin{cases} 0, & n - \text{for n even} \\ 1, & n - \text{for n odd} \end{cases}, \quad \sigma = \sqrt{2a\tau}; \quad \lambda = E\left(\frac{n}{2}\right); \ E(k) = \text{entier}(k);$$

$$(-1)!! = 1.$$

The algorithm for the approximate solution of this inverse problem is based on the solution of (5) for the forward problem and is extremely simple. We specify a sufficiently high order L and approximate the function $T(x, \tau)$ by the polynomial $T_P(x, \tau)$ with the necessary accuracy. Then the coefficients of the resulting polynomial are simultaneously the corresponding coefficients of the polynomial $P(\xi)$ with the necessary accuracy. Then the coefficients of the resulting polynomial are simultaneously the corresponding coefficients of the polynomial $P(\xi)$, completely determining it in the linear combination (4).

The uniqueness of the solution in the class of polynomials is obvious. We turn now to the solution of the following inverse problem. We replace T, f, and a by C, C_0, and D, respectively, adopting the standard notation of diffusion theory. Then Eq. (1) describes the free one-dimensional diffusion of a material in an unbounded space, while the quantity $\sigma = \sqrt{2D\tau}$ represents the mean square distance of particles which have undergone diffusion for a time τ from their initial position. We further assume that σ is not known but that, along with the distribution $C(x, \sigma)$, the value of the unknown function $C_0(\xi)$ is given at some point ξ_0 on the interval $(-\infty, \infty)$. The mathematical formulation of this problem is as follows: We are to find the solution of the integral equation

$$C(x,\ \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} C_0(\xi) \exp\left[-\frac{(\xi - x)^2}{2\sigma^2}\right] d\xi \tag{7}$$

with the unknown parameter σ under the condition

$$C_0(\xi_0) = C^{(0)}, \ \xi_0 \in (-\infty,\ \infty). \tag{8}$$

This unusual formulation of the inverse problem has found important applications in interpreting the diffuse halos which develop on flat and convex slopes above deposits of rare and nonferrous metals [5, 6]. In this case the known distribution $C(x, \sigma)$ of the chemical element in question in recent porous formations is used to predict the distribution of this element in early bedrock, $C_0(\xi)$. Sampling of a natural or artificial outcrop of this bedrock at some point yields the value of $C^{(0)}$. The parameter σ is not known to the investigator.

The solution of the problem is greatly complicated by the uncertainty regarding the parameter σ, which appears in Eq. (7) in a nonlinear manner. This circumstance makes it necessary to carry out a special study of the single-valuedness of the solution and, in particular, hinders a numerical solution of Eq. (7) by the known regularization methods. The iterative method for seeking a solution by the net method, with variation of σ over some plausible range, on the other hand, is extremely laborious. Furthermore, replacement of the original operator by an approximating operator introduces a large error into the result of the solution of the inverse problem.

We will also examine the single-valuedness of the solution of problem (7), (8) in the class of algebraic polynomials, and in this connection we will give an efficient algorithm for an approximate analytic solution for this problem based on the polynomial approximation (4) of the unknown function $C_0(\xi)$. At first we simply note that, without restriction (8), Eq. (7) has an infinite set of solutions. The two different polynomials

$$C_0^{(1)}(\xi) = \sum_{i=0}^{L_1} A_i^{(1)}(\sigma_1)\xi^i \text{and} C_0^{(2)}(\xi) = \sum_{i=0}^{L_2} A_i^{(2)}(\sigma_2)\xi^i \tag{9}$$

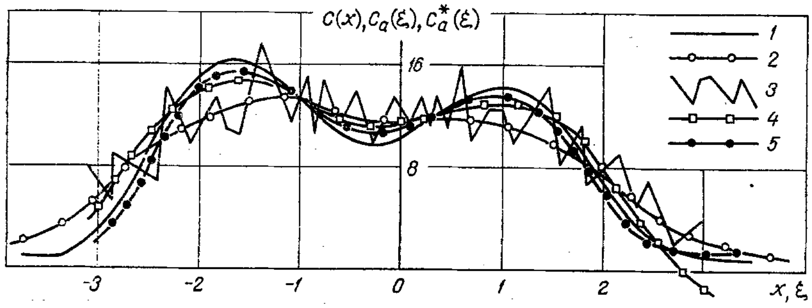

Fig. 1. Illustrative solution of problem (7), (8) (in arbitrary units). 1) Theoretical concentration distribution $C_0(\xi)$; 2) theoretical concentration distribution C(x); 3) "observed" distribution following concentrations $\tilde{C}(x)$; 4) algebraic polynomial $C_1^*(\xi)$ (σ = 0.52) which is the optimum approximation of the function $C_0(\xi)$; 5) trigonometric polynomial $C_2^*(\xi)$ (σ = 0.58) which is the optimum approximation of the function $C_0(\xi)$.

are mapped by integral transformation (7) into the same function C(x) if and only if

$$L_2 = L_1 = L,\ A_L^{(1)} = A_L^{(2)},\ A_{L-1}^{(1)} = A_{L-1}^{(2)},$$
$$A_j^{(1)} = A_j^{(2)} + \sum_{k=1}^{\mu} \binom{2k+j}{j} (2k-1)!!\,(A_{2k+j}^{(2)} \cdot \sigma_2^{2k} - A_{2k+j}^{(1)} \sigma_1^{2k}), \tag{10}$$
$$\mu = E\left(\frac{L-j}{2}\right),\ j = L-2,\ L-3,\ \ldots,\ 2,\ 1,\ 0.$$

Recurrence relations (10) determine a nondenumerable family of solutions of Eq. (7).

We now assume that $C_0(\xi)$ belongs to the class of algebraic polynomials and that the initial data C(x) and C^0 are not "burdened" with a random component. Without any loss of generality we can also assume that we know the degree L of polynomial $C_0(\xi)$. Specifying an arbitrary value $\sigma = \sigma_0 > 0$, we expand the function C(x) in terms of the components $\varphi_n(x, \sigma_0)$ ($n = \overline{0, L}$) without a residue, and we determine the unique solution $C_0(\xi, \sigma_0)$ of Eq. (7) for a fixed value of σ. In general, the curve $C_0(\xi, \sigma_0)$ does not pass through the point (ξ_0, $C^{(0)}$) and thus is not a solution of problem (7), (8); we denote this solution by $C_0(\xi, \sigma)$, and we denote the coefficients of the polynomial $C_0(\xi, \sigma)$ by $B_i(\sigma)$. Then we have the equation

$$\sum_{i=0}^{L} \xi_0^i (B_i(\sigma) - A_i(\sigma_0)) + C_0(\xi_0, \sigma_0) - C^{(0)} = 0. \tag{11}$$

Using recurrence relations (10) we can express the unknowns B_i in terms of σ and the known quantities $A_i(\sigma_0)$ and σ_0. Then Eq. (11) is an equation for the parameter σ and can be transformed after some manipulation to the form

$$\sum_{m=0}^{E\left(\frac{L}{2}\right)} W_m \sigma^{2m} = 0, \tag{12}$$

where

$$W_m = \begin{cases} \sum_{n=0}^{L} \xi_0^n (S_0^{(n)} - A_n(\sigma_0)) + C_0(\xi_0, \sigma_0) - C^{(0)}, & m = 0, \\ \sum_{n=0}^{L-2m} S_m^{(n)} \xi_0^n, & m \geqslant 1, \end{cases}$$

$$
S_t^{(n)} = \begin{cases} P_n, \ t=0, \\ P_{n+2t} \cdot \sum_{i=1}^{t} (-1)^i R_{t,i}, \ 1 \leqslant t \leqslant E(\nu)-1, \\ \left.\begin{array}{l} A_L \sum_{i=0}^{\nu-1} (-1)^i K_i, \ (L-n) - \text{even} \\ A_{L-1} \sum_{i=0}^{E(\nu)-1} (-1)^i K_i, \ (L-n) - \text{odd} \end{array}\right\} t = E(\nu), \end{cases}
$$

$$
K_i = \begin{cases} Q_{E(\nu),n}, \ i=0, \\ q_1(Q_{j_1,n}), \ i=1, \\ q_i(T_{i,2}(T_{i,3}(\ldots(T_{i,i-1}(T_{i,i}(Q_{j_i,n})))\ldots))), \ i>1, \end{cases}
$$

$$
R_{t,i} = \begin{cases} Q_{t,n}, \ i=1, \\ p_{t,2}(Q_{j_2,n}), \ i=2, \\ p_{t,i}(T_{i,3}(T_{i,4}(\ldots(T_{i,i-1}(T_{i,i}(Q_{j_i,n})))), \ i>2, \end{cases}
$$

Here $T_{i,s}$, $p_{t,i}$, and q_i are summation operators,

$$
T_{i,s}(u) = \sum_{j_s=1}^{j_{s-1}} Q_{j_{s-1}-j_s+1,n+2j_s+2i-2s} u,
$$

$$
p_{t,i}(u) = \sum_{j_2=1}^{t-i+1} Q_{t-i-j_2+2,n+2j_2+2i-4} u,
$$

$$
q_i(u) = \sum_{j_1=1}^{E(\nu)-i} Q_{E(\nu)-i-j_1+1,n+2j_1+2i-2} u,
$$

so that $R_{t,i}$ and K_i are finite chains of the enclosed sums, in which the upper limit on the summation in each sum is set by the running value of the summation index of the preceding sum:

$$
Q_{k,s} = \binom{2k+s}{s}(2k-1)!!, \quad P_s = A_s + \sum_{k=1}^{E\left(\frac{L-s}{2}\right)} Q_{k,s} A_{2k+s} \sigma_0^{2k}, \quad \nu = \frac{L-n}{2}.
$$

Equation (12) is algebraic. Let us determine its positive roots; there are no more than E(L/2) of them, and any one of them is the desired result. We substitute the values found for σ into recurrence relations (10) and find a finite number of solutions of problem (7), (8). Discarding some of these solutions on physical grounds [the concentration of the material over the interval $(-\infty, \infty)$ being steadied is nonnegative and does not exceed 100%], we find that in general there are $N \leq E(L/2)$ equiprobable solutions of Eq. (7) under condition (8).

Accordingly, even if we know the value of the desired concentration $C_0(\xi)$ at some point ξ_0, we do not eliminate the ambiguity from the solution of problem (7), (8). However, condition (8) does significantly restrict the spectrum of solutions of Eq. (7), permitting us to distinguish from the nondenumerable set of such solutions a finite number (perhaps only one).

It is not difficult to see that in addition to proving the boundedness of the solution of (7), (8) in the class of polynomials we have also studied an efficient algorithm for seeking an approximate analytic solution of this problem. This algorithm requires simply a single solution of Eq. (7); then the entire spectrum of allowed values of $C_0^*(\xi)$ is determined as consisting of the roots of an algebraic equation. To find the corresponding values of the approximating functions C(x) on the other hand, it is not necessary to solve Eq. (7). It is simply necessary to emphasize that in practice the function C(x) is usually specified in a discrete manner, so that the order L of the approximating polynomial $C_0^*(\xi)$ is bounded [if only by the number of points at which C(x) is measured]. Furthermore, the function C(x) contains a random component, and the desired distribution $C_0(\xi)$ is, generally speaking, not a polynomial. Hence we can draw the following conclusions:

1) Equation (12) may not contain solutions satisfactory for our purposes (N = 0). In this case we can choose that branch of the solution of Eq. (7) which is closest to $C^{(0)}$ at the point ξ_0 as the desired solution.

2) On the other hand, there may be more than one possible solution (N > 1). In this case a unique solution can be chosen by imposing, e.g., the requirement that the solution be smooth.

3) Since the class of algebraic polynomials does not exhaust the set of all continuous functions, the algorithm described here for approximately solving problem (6), (7) should be classified as one of the so-called algorithms for solving the inverse problem by the trial and error method; such algorithms permit us to seek at least one (but not necessarily all) of the solutions of the problem.

4) Since the inverse heat-conduction problem is incorrect in the Hadamard sense, an increase in the degree of the polynomial approximating the input function C(x) leads to the absence of a continuous dependence of the solution on the input data. In this case the resulting solution must be regularized, as shown, e.g., by Alifanov [1].

Without any particular difficulty the results obtained above can be generalized to the cases of two-dimensional and three-dimensional inverse problems in unbounded and semibounded spaces. Finally, it should be noted that approximate analytic solutions of these problems can be constructed by adopting as a basis the system of trigonometric functions $\Psi = \{1, \{\sin i\xi, \cos i\xi\}_{i=1}^{\infty}\}$, which permit an approximation of any continuous functions on an integral of length 2π.

These algorithms were used in an ALGOL program for a BÉSM-4 computer and tested for several models. The solutions for small values of L were found to be very stable with respect to errors in the initial data. This circumstance is illustrated by the following typical example of the solution of inverse problem (7), (8) through the use of both polynomial and trigonometric approximations.

The theoretical distribution $C_0(\xi)$ is part of a Fourier series defined on the interval $(-\pi, \pi)$. The values of C(x) due to this distribution and the value of $\sigma = 0.75$ were calculated at the mesh points of a uniform net on the basis of Eq. (7) and high-precision quadratic equations. The influence of a random component on C(x) was simulated by imposing log-normally distributed random numbers, taken from [7], on the actual values C(x). Then the computer searched for $C_0^*(\xi)$ in the classes of trigonometric and algebraic polynomials by the algorithm described above. The parameter was determined from the condition $C_0(2.65) = 0.8$ arbitrary unit. The optimum number of parameters of the approximating polynomials was chosen on the basis of the criterion proposed in [8]. Figure 1 clearly demonstrates that the selected function $C_1^*(\xi)$ and $C_2^*(\xi)$ agree well with the desired function.

NOTATION

a	is the thermal diffusivity;
T	is the temperature;
C	is the concentration;
D	is the diffusion coefficient;
(x)	is the initial temperature distributions;
C_0	is the initial concentration distribution;
x	is the coordinate;
τ	is the time.

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